Global Min-cuts

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Defn: For an undirected graph $G = (V, E)$, for cut $(S, V \setminus S)$, let $\delta(S) = \{e = (v,w) : |e \cap S| = 1\}$ (# edges across cut).

Problem: Given an undirected graph $G = (V, E)$, find cut $(S, V \setminus S)$ of minimum cardinality for $|\delta(S)|$.

(earlier: s–t mincut. given directed graph $G = (V, E)$, capacities $c_e \in \mathbb{Z}_+$, spl. vertices s, t, find s–t min-cut of minimum capacity)

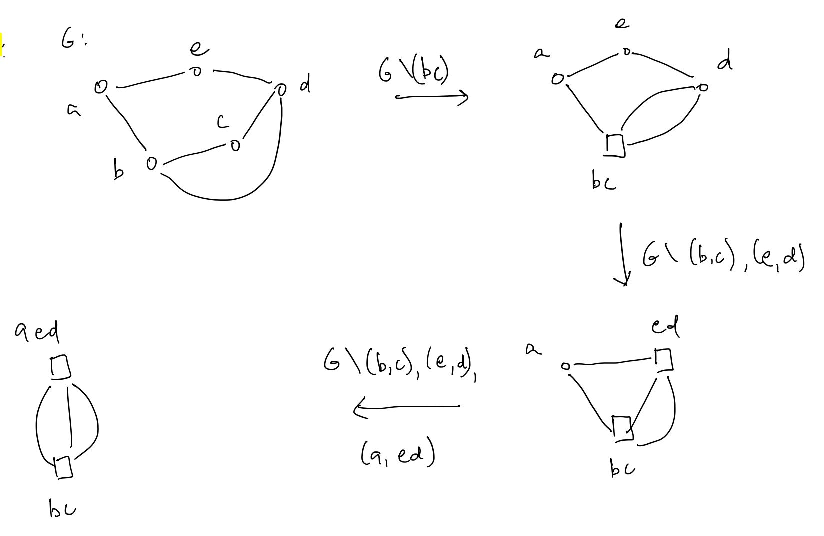
We can find global min-cut using $(n-1)$ max-flow computations (prove yourself).
By the push-relabel algo, this gives an $O(n) \times O(n^2 \sqrt{m})$ time algo to compute the global min-cut.

We will instead look at randomized algorithms, which give a global min-cut with high probability.

Contraction: G/e is the multigraph (→ allows parallel edges, but no self-loops) obtained by "contracting" edge $e = (v, w)$

i.e., $G/e = (V', E')$ where:
- $V' = (V \setminus \{v, w\}) \cup \{z\}$
- E' consists of all edges in E not incident on $\{v, w\}$.
  Further, for all $x \neq v, w$,
  replace all (x, v) edges by (x, z) edges (could be more than one!)
  replace all (x, w) edges by (x, z) edges

Example:



Claim: Say multigraph G has min-cut $(S, V \setminus S)$ of size k (i.e., $|\delta(S)| = k$). Then the minimum degree $\geq k$, and hence $|E| \geq \frac{k}{2}|V|$.

(easy)

Now consider the cuts in (multigraphs) G & $G \setminus e = (V', E')$ ($e = (v, w)$)
- any cut in $G \setminus e$ (say $S', V' \setminus S'$ with $(v,w) \in S'$) is also a cut $(S, V \setminus S)$ in G, & $|\delta(S')| = |\delta(S)|$.

Claim: thus size of min-cut in $G \setminus e$ $\geq$ size of min-cut in G.

from earlier claim: # edges in $G \setminus e \geq \frac{k}{2}|V'|$

- a cut in G is also a cut in $G \setminus e$, unless the cut separates (v, w).

If we do $n-2$ edge contractions, we are left with 2 supernodes $(X, V \setminus X)$.
Suppose each edge for contraction is chosen uniformly from the set of remaining edges. What is the probability that $(X, V \setminus X)$ is a min-cut in graph G?

Fix a min-cut $(S, V \setminus S)$. Let $k = |\delta(S)|$

Lemma: After $n-2$ edge contractions, with each successive edge chosen u.a.r. from the remaining edges, the probability that $(S, V \setminus S)$ are the two remaining edges is at least $1/\binom{n}{2}$.

Proof: Let $e_1, e_2, \ldots, e_{n-2}$ be the edges chosen.
$|\delta(S)| = k$. Hence $\Pr(e_1 \in \delta(S)) \leq \frac{k}{kn/2} = \frac{2}{n}$ (from first claim.)

$\Pr(e_2 \in \delta(S) \mid e_1 \notin \delta(S)) \leq \frac{k}{k(n-1)/2} = \frac{2}{n-1}$ (2nd claim)

in general,
$\Pr(e_i \in \delta(S) \mid e_1, \ldots, e_{i-1} \notin \delta(S)) \leq \frac{k}{\frac{k}{2}(n-i+1)} = \frac{2}{n-i+1}$

Hence: $\Pr(e_1 \notin \delta(S) \wedge e_2 \notin \delta(S) \wedge \ldots \wedge e_{n-2} \notin \delta(S))$

$= \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right) \cdots \left(1 - \frac{2}{3}\right)$ (v. small → … ← v. large)

$= \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right)\left(\frac{n-4}{n-2}\right) \cdots \left(\frac{2}{4}\right)\left(\frac{1}{3}\right)$

$= \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}}$ ∎

ok, so now if we repeat this $2n^2 \log n$ times,

$\Pr(\text{failure}) \leq \left(1 - \frac{1}{\binom{n}{2}}\right)^{2n^2 \log n}$

$\leq \left(1 - \frac{1}{n^2}\right)^{2n^2 \log n} \leq e^{-2n^2 \log n / n^2}$

$= \frac{1}{n^2}$

so the algorithm obtains a min-cut w.h.p.

(This is called Karger's algorithm)

Karger-GMC ($G = (V, E)$):

```
s ← ∞
for i = 1 ... 2n² log n
    H ← G
    for j = 1 ... n-2
        choose an edge e = (v,w) in H u.a.r.
        H ← H/e
    s ← min(s, |E(H)|)
```

With some care, the contraction can be done in $O(n)$ time, so this gives an $O(n^4 \log n)$ time algorithm.

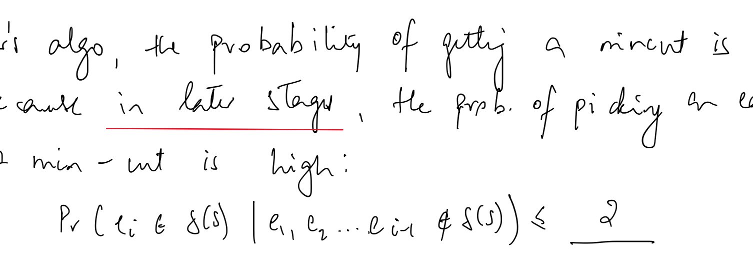
The Karger–Stein algorithm:

In Karger's algo, the probability of getting a mincut is low because in later stages, the prob. of picking an edge from a min-cut is high:

$\Pr(e_i \in \delta(S) \mid e_1, e_2 \ldots e_{i-1} \notin \delta(S)) \leq \frac{2}{n-i+1}$

$\Rightarrow \Pr(e_1, \ldots, e_i \notin \delta(S)) \geq \frac{\binom{n-i}{2}}{\binom{n}{2}} \geq \frac{1}{2} \Rightarrow 2(n-i)^2 \geq n^2$ $(n - i \geq n/\sqrt{2})$

Instead of running the entire algo n^2 times, suppose we just run the "later" stages multiple times?



i.e., Karger-Stein GMC ($G = (V, E)$)

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if |V| ≤ 6, return min-cut
s ← ∞
for i = 1, 2
    H ← G
(*) run Karger GMC for n(1 - 1/√2) steps
        (# vertices remaining = n/√2)
    run Karger-Stein GMC on resulting graph
    s ← min(s, value returned)
```

Let $P(n)$ be the probability of success with n nodes, i.e., Prob that KSGMC returns a min-cut in a graph w/ n nodes.
Probability that a single run (i.e., for $i = 1$ in algo) succeeds is $\frac{1}{2} \times P\left(\frac{n}{\sqrt{2}}\right)$ (← recursive step; $\frac{1}{2}$ ↑ prob that (*) does not destroy min-cut)

Probability that at least one of the two runs succeeds is $1 - \left(1 - \frac{1}{2}P\left(\frac{n}{\sqrt{2}}\right)\right)^2$

Claim: $P(n) \geq \frac{1}{\log_2 n}$ (verify yourself)

Thus Karger Stein GMC succeeds w.p. $\geq 1/\log_2 n$
So if we run it $2\log^2 n$ times, probability of success is

$\geq 1 - \left(1 - \frac{1}{\log n}\right)^{2\log^2 n} = 1 - \frac{1}{n^2}$

Time taken is $2\log^2 n \; T(n)$, where
$T(n) = O(n^2) + 2T(n/\sqrt{2})$
$= O(n^2 \log n)$
hence total run time is $O(n^2 \log^3 n)$.